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A System of Homogeneous Spherical Harmonics

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In a previous note [1] under this title there was given a method for constructing a sequence of solid spherical harmonics $Y_s(\mathbf{x})$, the first few of which are

$$
\begin{aligned}
Y_i &= x_i, \\
Y_{ij} &= x_i x_j - \frac{1}{3} x^2 \delta_{ij}, \\
Y_{ijk} &= x_i x_j x_k - \frac{3}{5} x^2 x(_i \delta_{jk}), \qquad (1) \\
Y_{ijkl} &= x_i x_j x_k x_l - \frac{6}{7} x^2 x(_i x_j \delta_{kl}) + \frac{3}{35} x^4 \delta(_{ij} \delta_{kl}),
\end{aligned}
$$

where

$$x^2 = x_1{}^2 + x_2{}^2 + x_3{}^2, \qquad (2)$$

the subscripts i, j, k, ... may each have any of the values 1, 2, or 3, and the parentheses enclosing s subscripts means to sum over the $s!$ terms obtained by all permutations of the s subscripts, and then divide by $s!$ These polynomials have more recently appeared in researches in statistical mechanics [2] and in the quantum theory of angular momentum [3].

A second method for obtaining these polynomials is mentioned by Edmonds [3] who states that the polynomials of degree s may be obtained by subtracting from the tensor $\mathbf{x}^s$ all quantities which are invariant under orthogonal transformations. The tensor $\mathbf{Y}_s(\mathbf{x})$ thus obtained is completely symmetric and reduces to zero when contracted on any pair of indices.

The two methods mentioned above for obtaining the solid spherical harmonics $Y_s(\mathbf{x})$ and the tensors $\mathbf{Y}_s(\mathbf{x})$ do not afford a general representation for all s and, while straightforward in principle, become prohibitively tedious as s increases. In this note we give a general representation in terms of the Legendre polynomials $P_s(\mu)$ from which, for any given s, $Y_s(\mathbf{x})$ is readily obtainable.

The Legendre polynomial $P_s(\hat{\mathbf{r}} \cdot \hat{\mathbf{x}})$, where

$$P_s(\mu) = \frac{1}{2^s s!} \frac{d_s}{d\mu^s} (\mu^2 - 1)^s \tag{3}$$

and $\hat{\mathbf{r}}$ and $\hat{\mathbf{x}}$ are unit vectors parallel to $\mathbf{r}$ and $\mathbf{x}$, respectively, is a surface spherical harmonic of degree s in $\hat{\mathbf{x}}$. The solid spherical harmonic $(rx)^s P_s(\hat{\mathbf{r}} \cdot \hat{\mathbf{x}})$ is a homogeneous polynomial of degree s in $\mathbf{r}$ as well as in $\mathbf{x}$. Since $(rx)^s P_s(\hat{\mathbf{r}} \cdot \hat{\mathbf{x}})$ is invariant under orthogonal transformations and $\mathbf{r}^s$ is a tensor of order s, we may write

$$(rx)^s P_s(\hat{\mathbf{r}} \cdot \hat{\mathbf{x}}) = \frac{(2s)!}{2^s s! s!} \mathbf{r}^s \cdot \mathbf{T}_s(\mathbf{x}) \tag{4}$$

where $\mathbf{T}_s(\mathbf{x})$ is a tensor of order s with components which are homogeneous polynomials of degree s in $\mathbf{x}$. As the arbitrary numerical factor in the right hand member of (4) we have chosen the coefficient $(2s!)/2^s s! s!$ of μ^s in $P_s(\mu)$.

Tensors $\mathbf{T}_s(\mathbf{x})$ of order s are not uniquely determined by (4) above for, given any tensor $\mathbf{T}_s(\mathbf{x})$ satisfying (4), we can obtain another such one by symmetrizing in any pair of indices. It is quite readily seen, however, that the desired tensor $\mathbf{Y}_s(\mathbf{x})$ is the only completely symmetric tensor such that

$$(rx)^s P_s(\hat{\mathbf{r}} \cdot \hat{\mathbf{x}}) = \frac{(2s)!}{2^s s! s!} \mathbf{r}^s \cdot \mathbf{Y}_s(\mathbf{x}). \tag{5}$$

$\mathbf{Y}_s(\mathbf{x})$ is, therefore, obtainable by completely symmetrizing any tensor $\mathbf{T}_s(\mathbf{x})$ satisfying (4). Formally, the symmetrization is readily accomplished by the subscript notation used in (1).

The method may be adequately illustrated with $s = 4$. Starting with

$$P_4(\mu) = \frac{35}{8} \mu^4 - \frac{15}{4} \mu^2 + \frac{3}{8}, \tag{6}$$

we write

$$\begin{aligned} (rx)^4 P_4(\hat{\mathbf{r}} \cdot \hat{\mathbf{x}}) &= \frac{35}{8} \left\{ (\mathbf{r} \cdot \mathbf{x})^4 - \frac{6}{7} r^2 x^2 (\mathbf{r} \cdot \mathbf{x})^2 + \frac{3}{35} r^4 x^4 \right\} \\ &= \frac{35}{8} \mathbf{r}^4 \cdot \mathbf{T}_4(\mathbf{x}), \end{aligned} \tag{7}$$

where $\mathbf{T}_4(\mathbf{x})$ has components

$$T_{ijkl} = x_i x_j x_k x_l - \frac{6}{7} x^2 x_i x_j \delta_{kl} + \frac{3}{35} x^4 \delta_{ij} \delta_{kl}. \tag{8}$$

From T_{ijkl} we readily write down the completely symmetrical Y_{ijkl}, as given in (1).

References

1. Ikenberry, E. *Am. Math. Monthly* **62**, 719–21 (1955).
2. Ikenberry, E., and Truesdell, C. *J. Rational Mech. Analysis* **5**, 1–54 (1956).
3. Edmonds, A. R. "Angular Momentum in Quantum Mechanics," p. 68. Princeton Univ. Press, Princeton, New Jersey, 1957.